CS-GY 6763: Lecture 10
Randomized numerical linear algebra, ϵ-net arguments.

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ANNOUNCEMENTS

- HW3 Due tonight
- HW4 out by tomorrow.
- **Final Exam:** In class, on the last class Monday May 9th (not during scheduled final slot Tueday May 10th!)
- Reading Group this Thursday: Atsushi will discuss the Contextual Bandits problem. Dennis and Jesse are Discussion leaders (presenters from last week).
- My office hours, moved to 4:30-5:30 Wednesday (just for this week).

RANDOMIZED NUMERICAL LINEAR ALGEBRA

Today: randomized algorithms for sketching (compressing) matrices

- Given a dense $n \times n$ matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$.
- Computing top eigenvectors takes $\approx O(n^2/\sqrt{\epsilon})$ time (via power method/Krylov methods from last class).

If someone asked you to speed this up and return <u>approximate</u> top eigenvectors, what could you do?

What about approximately solving the regression problem:

$\hat{x}$ $\quad f(\hat{x}) < f(x) + \epsilon$

$$\min_x f(x) = \min_x \|\mathbf{A}x - b\|_2$$

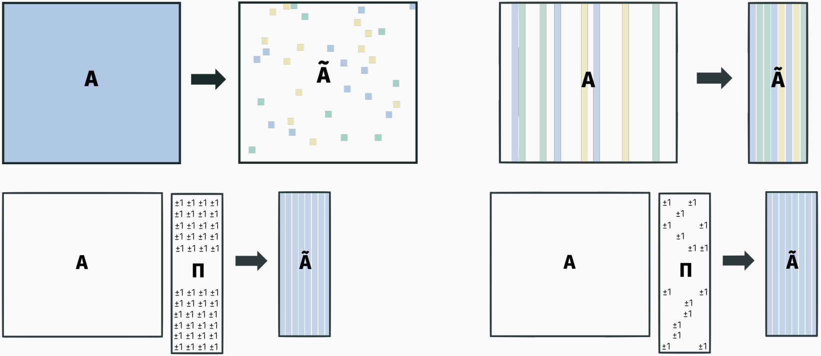
$|A\hat{x} - b|_2$

RANDOMIZED NUMERICAL LINEAR ALGEBRA

Main idea: If you want to compute singular vectors, multiply two matrices, solve a regression problem, etc.:

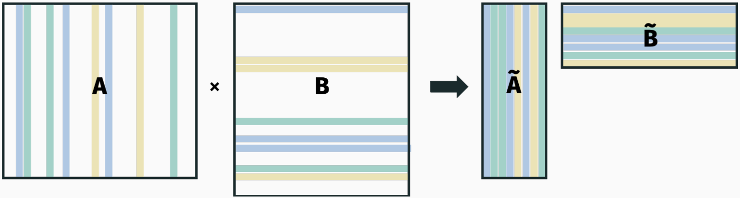
1. Compress your matrices using a randomized method (e.g. subsampling).
2. Solve the problem on the smaller or sparser matrix.
   - $\tilde{\mathbf{A}}$ called a "sketch" or "coreset" for $\mathbf{A}$.

Sketch n' Solve

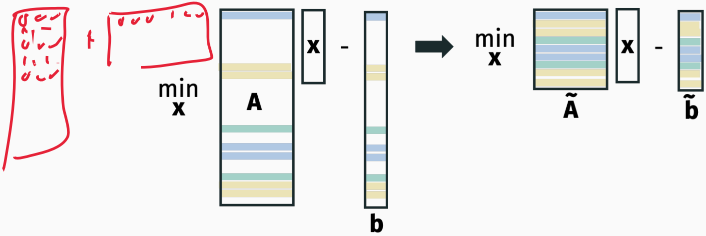


RANDOMIZED NUMERICAL LINEAR ALGEBRA

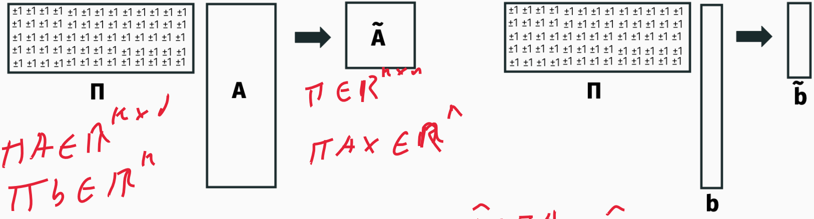
Approximate matrix multiplication:



Approximate regression:



Randomized approximate regression using a JL Matrix:



$\Pi A \in \mathbb{R}^{k \times d}$

$\Pi b \in \mathbb{R}^{k}$

$\Pi \in \mathbb{R}^{k \times n}$

$\Pi A x \in \mathbb{R}^{k}$

$\hat{A} = \Pi A$ $\quad \hat{b} = \Pi b$

$\min_x |\tilde{A}x - \tilde{b}|_2^2$

Input: $\mathbf{A} \in \mathbb{R}^{n \times d}$, $\mathbf{b} \in \mathbb{R}^n$.

Goal: Let $\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$, and
$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{\Pi}\mathbf{A}\mathbf{x} - \mathbf{\Pi}\mathbf{b}\|_2^2$

$$\text{Want: } \|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2^2 \leq (1 + O(\epsilon)) \|\mathbf{A}\mathbf{x}^* - \mathbf{b}\|_2^2$$

If $\mathbf{\Pi} \in \mathbb{R}^{m \times n}$, how large does m need to be? Is it even clear this should work as $m \to \infty$?

TARGET RESULT

Theorem (Randomized Linear Regression)

Let $\mathbf{\Pi}$ *be a properly scaled JL matrix (random Gaussian, sign, sparse random, etc.) with* $m = O\left(\frac{d}{\epsilon^2}\right)$ *rows.*[1] *Then with probability* $9/10$, *for any* $\mathbf{A} \in \mathbb{R}^{n \times d}$ *and* $\mathbf{b} \in \mathbb{R}^n$,

$$\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}^* - \mathbf{b}\|_2^2$$

where $\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{\Pi}\mathbf{A}\mathbf{x} - \mathbf{\Pi}\mathbf{b}\|_2^2$.

$f(\tilde{x}) < (1+\epsilon) f(x^*)$

$\|\Pi \underbrace{(Ax-b)}\|_2 \approx \|Ax-b\|_2^2$

$x \in \mathbb{R}^d$

Prob of success $= 1 - n \cdot \delta$

$\delta :=$ failure n event

[1] This can be improved to $O(d/\epsilon)$ with a tighter analysis

PLAN

- Prove this theorem using an ϵ-net argument, which is a popular technique for applying our standard concentration inequality + union bound argument to an infinite number of events.
- These sort of arguments appear all the time in theoretical algorithms and ML research, so this lecture is as much about the technique as the final result.
- You will need to use and ϵ-net argument to prove a matrix concentration inequality on your problem set.

SKETCHED REGRESSION

Claim: Suffices to prove that for all $\mathbf{x} \in \mathbb{R}^d$,

$$(1-\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 \leq \underbrace{\|\mathbf{\Pi}\mathbf{A}\mathbf{x}-\mathbf{\Pi}\mathbf{b}\|_2^2}_{\tilde{f}(x)} \leq (1+\epsilon)\underbrace{\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2}_{f(x)}$$

$\tilde{x} = \arg\min_x \tilde{f}(x)$ $\qquad$ $x^* = \arg\min_x f(x)$

(1) $\quad \underbrace{\tilde{f}(\tilde{x})}_{} < \tilde{f}(x^*) < \underbrace{(1+\epsilon)\, f(x^*)}_{}$

(2) $\quad f(\tilde{x}) < (1+\epsilon)\underbrace{\tilde{f}(\tilde{x})}_{} < (1+\epsilon)(1+\epsilon) f(x^*)$

$f(\tilde{x}) < (1+\epsilon)^2 f(x^*)$

DISTRIBUTIONAL JOHNSON-LINDENSTRAUSS REVIEW

Lemma (Distributional JL)

If $\mathbf{\Pi}$ is chosen to a properly scaled random Gaussian matrix, sign matrix, sparse random matrix, etc., with $\underbrace{O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)}$ rows then for any fixed $\mathbf{y}$,

$$(1-\epsilon)\|\mathbf{y}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{y}\|_2^2 \leq (1+\epsilon)\|\mathbf{y}\|_2^2$$

with probability $(1-\delta)$.

$y = Ax - b$

Corollary: For any fixed $\mathbf{x}$, with probability $(1-\delta)$,

$$\underbrace{(1-\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{A}\mathbf{x}-\mathbf{\Pi}\mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2.}$$

FOR ANY TO FOR ALL

How do we go from "<u>for any fixed $\mathbf{x}$</u>" to "<u>for all $\mathbf{x} \in \mathbb{R}^d$</u>".

This statement requires establishing a Johnson-Lindenstrauss type bound for an <u>infinity</u> of possible vectors $(\mathbf{A}\mathbf{x} - \mathbf{b})$, which can't be tackled directly with a union bound argument.

FOR ANY TO FOR ALL

How do we go from "for any fixed $\mathbf{x}$" to "for all $\mathbf{x} \in \mathbb{R}^d$".

This statement requires establishing a Johnson-Lindenstrauss type bound for an infinity of possible vectors $(\mathbf{A}\mathbf{x} - \mathbf{b})$, which can't be tackled directly with a union bound argument.

Note: all vectors of the form $(\mathbf{A}\mathbf{x} - \mathbf{b})$ lie in a low dimensional subspace: spanned by $d + 1$ vectors, where $\mathbf{A} \in \mathbb{R}^{n \times d}$.

Even though the set is infinite, it is only $O(d)$-dimensional instead of $O(n)$.

SUBSPACE EMBEDDINGS

Theorem (Subspace Embedding from JL)

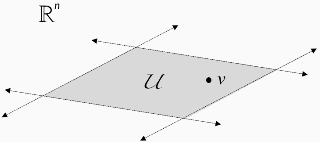
Let $\mathcal{U} \subset \mathbb{R}^n$ be a d-dimensional linear subspace in $\mathbb{R}^n$. If $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is chosen from any distribution $\mathcal{D}$ satisfying the Distributional JL Lemma, then with probability $1 - \delta$,

$v = Ax - b$

$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\Pi\mathbf{v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2$$

$\Pi(Ax - b)$

for all $\mathbf{v} \in \mathcal{U}$, as long as $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$[2].



[2] It's possible to obtain a slightly tighter bound of $O\left(\frac{d+\log(1/\delta)}{\epsilon^2}\right)$. It's a nice challenge to try proving this.

SUBSPACE EMBEDDING TO APPROXIMATE REGRESSION

Corollary: If we choose $\mathbf{\Pi}$ and properly scale, then with $O\left(d/\epsilon^2\right)$ rows,

$$(1-\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{A}\mathbf{x}-\mathbf{\Pi}\mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2$$

for all $\mathbf{x}$ and thus

$$\|\mathbf{A}\tilde{\mathbf{x}}-\mathbf{b}\|_2^2 \leq (1+O(\epsilon))\min_{\mathbf{x}}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2.$$

I.e., our main theorem is proven.

Proof: Apply Subspace Embedding Thm. to the $(d+1)$ dimensional subspace spanned by $\mathbf{A}$'s d columns and $\mathbf{b}$. Every vector $\mathbf{A}\mathbf{x}-\mathbf{b}$ lies in this subspace.

SUBSPACE EMBEDDINGS

Theorem (Subspace Embedding from JL)

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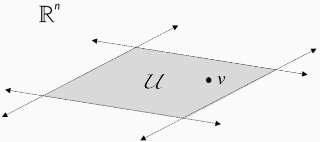
$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2 \tag{1}$$

for all $\mathbf{v} \in \mathcal{U}$, as long as $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$



Subspace embeddings have tons of other applications!

SUBSPACE EMBEDDING PROOF

$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2 \tag{2}$$

First Observation: The theorem holds as long as (2) holds for all $\mathbf{w}$ on the unit sphere in $\mathcal{U}$. Denote the sphere $S_{\mathcal{U}}$:

$$S_{\mathcal{U}} = \{\mathbf{w} \mid \mathbf{w} \in \mathcal{U} \text{ and } \|\mathbf{w}\|_2 = 1\}.$$

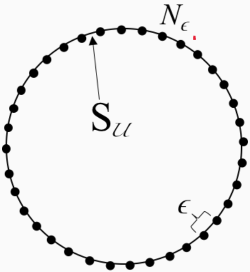
Follows from linearity: Any point $\mathbf{v} \in \mathcal{U}$ can be written as $c\mathbf{w}$ for some scalar c and some point $\mathbf{w} \in S_{\mathcal{U}}$.

$v = c\, w$

- If $(1-\epsilon)\|\mathbf{w}\|_2 \leq \|\mathbf{\Pi w}\|_2 \leq (1+\epsilon)\|\mathbf{w}\|_2$.
- then $c(1-\epsilon)\|\mathbf{w}\|_2 \leq c\|\mathbf{\Pi w}\|_2 \leq c(1+\epsilon)\|\mathbf{w}\|_2$,
- and thus $(1-\epsilon)\|c\mathbf{w}\|_2 \leq \|\mathbf{\Pi} c\mathbf{w}\|_2 \leq (1+\epsilon)\|c\mathbf{w}\|_2$.

SUBSPACE EMBEDDING PROOF

Intuition: There are not too many "different" points on a d-dimensional sphere:



N_ϵ is called an "ϵ"-net.

If we can prove

$$(1-\epsilon)\|\mathbf{w}\|_2 \leq \|\Pi\mathbf{w}\|_2 \leq (1+\epsilon)\|\mathbf{w}\|_2$$

for all points $\mathbf{w} \in N_\epsilon$, we can hopefully extend to all of $S_{\mathcal{U}}$.

Lemma (ϵ-net for the sphere)

For any $\epsilon \leq 1$, there exists a set $N_\epsilon \subset S_{\mathcal{U}}$ with $\underbrace{|N_\epsilon| = \left(\frac{4}{\epsilon}\right)^d}$ such that $\forall \mathbf{v} \in S_{\mathcal{U}}$,

$$\min_{\mathbf{w} \in N_\epsilon} \|\mathbf{v} - \mathbf{w}\| \leq \epsilon.$$

Take this claim to be true for now: we will prove later.

SUBSPACE EMBEDDING PROOF

$\left(\frac{4}{\epsilon}\right)^d \frac{1}{\delta}$

1. Preserving norms of all points in net N_ϵ.

$1 - \delta'$

Set $\delta' = \left(\frac{\epsilon}{4}\right)^d \cdot \delta$. By a union bound, with probability $1 - \delta$, for all $\mathbf{w} \in N_\epsilon$,

$|N_\epsilon| \cdot \delta' < \delta$

$$(1-\epsilon)\|\mathbf{w}\|_2 \leq \|\Pi \mathbf{w}\|_2 \leq (1+\epsilon)\|\mathbf{w}\|_2.$$

as long as Π has $O\left(\frac{\log(1/\delta')}{\epsilon^2}\right) = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$ rows.

$\log\left(\frac{1}{\delta'}\right) = d \log\left(\frac{4}{\epsilon}\right) + \log\left(\frac{1}{\delta}\right)$

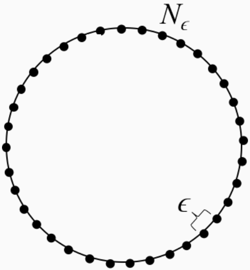
SUBSPACE EMBEDDING PROOF

2. Writing any point in sphere as linear comb. of points in N_ϵ.

For some $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2 \ldots \in N_\epsilon$, any $\mathbf{v} \in S_{\mathcal{U}}$, can be written:

$$\mathbf{v} = \mathbf{w}_0 + c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots$$

for constants $c_1, c_2, \ldots$ where $|c_i| \leq \epsilon^i$.



$|v - w_0|_2 = |r_0|_2 < \epsilon$

$c_1 = |r_0|_2 < \epsilon$

$|\underbrace{r_0 - c_1 w_1}_{r_1}| < \epsilon c_1 < \epsilon^2$

$c_2 = |r_1|$

$|r_1 - c_2 w_2| < \epsilon c_2 < \epsilon^3$

3. Preserving norm of v.

Applying triangle inequality, we have

$$
\begin{aligned}
\|\boldsymbol{\Pi}\mathbf{v}\|_2 &= \|\boldsymbol{\Pi}\mathbf{w}_0 + c_1\boldsymbol{\Pi}\mathbf{w}_1 + c_2\boldsymbol{\Pi}\mathbf{w}_2 + \ldots\| \\
&\leq \|\boldsymbol{\Pi}\mathbf{w}_0\| + \epsilon\|\boldsymbol{\Pi}\mathbf{w}_1\| + \epsilon^2\|\boldsymbol{\Pi}\mathbf{w}_2\| + \ldots \\
&\leq (1+\epsilon) + \epsilon(1+\epsilon) + \epsilon^2(1+\epsilon) + \ldots \\
&\leq 1 + O(\epsilon).
\end{aligned}
$$

3. Preserving norm of v.

Similarly,

$$\begin{aligned}
\|\mathbf{\Pi v}\|_2 &= \|\mathbf{\Pi w}_0 + c_1\mathbf{\Pi w}_1 + c_2\mathbf{\Pi w}_2 + \ldots\| \\
&\geq \|\mathbf{\Pi w}_0\| - \epsilon\|\mathbf{\Pi w}_1\| - \epsilon^2\|\mathbf{\Pi w}_2\| - \ldots \\
&\geq (1-\epsilon) - \epsilon(1+\epsilon) - \epsilon^2(1+\epsilon) - \ldots \\
&\geq 1 - O(\epsilon).
\end{aligned}$$

SUBSPACE EMBEDDING PROOF

So we have proven

$$(1 - O(\epsilon))\,\|\mathbf{v}\|_2 \leq \|\mathbf{\Pi v}\|_2 \leq (1 + O(\epsilon))\,\|\mathbf{v}\|_2$$

for all $\mathbf{v} \in S_{\mathcal{U}}$, which in turn implies,

$$(1 - O(\epsilon))\,\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi v}\|_2^2 \leq (1 + O(\epsilon))\,\|\mathbf{v}\|_2^2$$

Adjusting ϵ proves the Subspace Embedding theorem.

SUBSPACE EMBEDDINGS

Theorem (Subspace Embedding from JL)

Let $\mathcal{U} \subset \mathbb{R}^n$ be a d-dimensional linear subspace in $\mathbb{R}^n$. If $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is chosen from any distribution $\mathcal{D}$ satisfying the Distributional JL Lemma, then with probability $1 - \delta$,

$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2 \tag{3}$$

for all $\mathbf{v} \in \mathcal{U}$, as long as $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$

Subspace embeddings have many other applications!

For example, if $m = O(k/\epsilon)$, $\mathbf{\Pi A}$ can be used to compute an approximate partial SVD, which leads to a $(1+\epsilon)$ approximate low-rank approximation for $\mathbf{A}$.

ϵ-NET FOR THE SPHERE

Lemma (ϵ-net for the sphere)

For any $\epsilon \leq 1$, there exists a set $N_\epsilon \subset S_{\mathcal{U}}$ with $|N_\epsilon| = \left(\frac{4}{\epsilon}\right)^d$ such that $\forall \mathbf{v} \in S_{\mathcal{U}}$,

$$\min_{\mathbf{w} \in N_\epsilon} \|\mathbf{v} - \mathbf{w}\| \leq \epsilon.$$

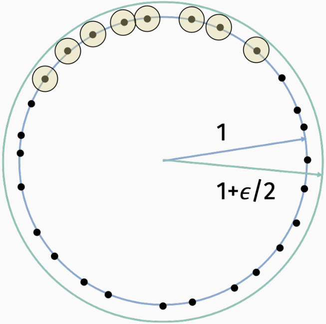
Imaginary algorithm for constructing N_ϵ:

- Set $N_\epsilon = \{\}$
- While such a point exists, choose an arbitrary point $\mathbf{v} \in S_{\mathcal{U}}$ where $\nexists \mathbf{w} \in N_\epsilon$ with $\|\mathbf{v} - \mathbf{w}\| \leq \epsilon$. Set $N_\epsilon = N_\epsilon \cup \{\mathbf{w}\}$.

After running this procedure, we have $N_\epsilon = \{\mathbf{w}_1, \ldots, \mathbf{w}_{|N_\epsilon|}\}$ and $\min_{\mathbf{w} \in N_\epsilon} \|\mathbf{v} - \mathbf{w}\| \leq \epsilon$ for all $\mathbf{v} \in S_{\mathcal{U}}$ as desired.

ϵ-NET FOR THE SPHERE

How many steps does this procedure take?



Can place a ball of radius $\epsilon/2$ around each $\mathbf{w}_i$ without intersecting any other balls. All of these balls live in a ball of radius $1 + \epsilon/2$.

ϵ-NET FOR THE SPHERE

Volume of d dimensional ball of radius r is

$$\mathrm{vol}(d, r) = c \cdot r^d,$$

where c is a constant that depends on d, but not r. From previous slide we have:

$$\begin{aligned} \mathrm{vol}(d, \epsilon/2) \cdot |N_\epsilon| &\leq \mathrm{vol}(d, 1 + \epsilon/2) \\ |N_\epsilon| &\leq \frac{\mathrm{vol}(d, 1 + \epsilon/2)}{\mathrm{vol}(d, \epsilon/2)} \\ &\leq \left(\frac{1 + \epsilon/2}{\epsilon/2}\right)^d \leq \left(\frac{4}{\epsilon}\right)^d \end{aligned}$$

TIGHTER BOUND

You can actually show that $m = O\left(\frac{d+\log(1/\delta)}{\epsilon^2}\right)$ suffices to be a d dimensional subspace embedding, instead of the bound we proved of $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$.

The trick is to show that a constant factor net is actually all that you need instead of an ϵ factor.

RUNTIME CONSIDERATION

For $\epsilon, \delta = O(1)$, we need $\mathbf{\Pi}$ to have $m = O(d)$ rows.

$Ax = b$

$x = A^{-1} b$

- Cost to solve $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$:
  - $O(nd^2)$ time for direct method. Need to compute $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.
  - $O(nd) \cdot (\#\text{ of iterations})$ time for iterative method (GD, AGD, conjugate gradient method).
- Cost to solve $\|\mathbf{\Pi}\mathbf{A}\mathbf{x} - \mathbf{\Pi}\mathbf{b}\|_2^2$:
  - $O(d^3)$ time for direct method.
  - $O(d^2) \cdot (\#\text{ of iterations})$ time for iterative method.

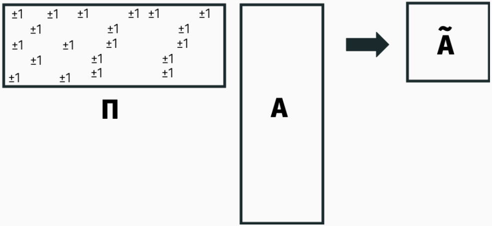
ΠA $\qquad$ $\Pi \in \mathbb{R}^{d \times d}$

$O(nd^2)$

RUNTIME CONSIDERATION

But time to compute $\mathbf{\Pi A}$ is an $(m \times n) \times (n \times d)$ matrix multiply: $O(mnd) = O(nd^2)$ time!

Goal: Develop faster Johnson-Lindenstrauss projections.



Typically using sparse and structured matrices.

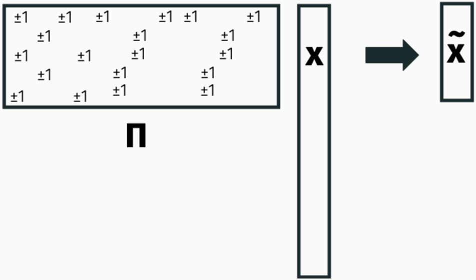
We will describe a construction where $\mathbf{\Pi A}$ can be computed in $O(nd \log n)$ time.

After the break: Super-Fast JL Projections

RETURN TO SINGLE VECTOR PROBLEM

Goal: Develop methods that reduce a vector $\mathbf{x} \in \mathbb{R}^n$ down to $m \approx \frac{\log(1/\delta)}{\epsilon^2}$ dimensions in $o(mn)$ time and guarantee:

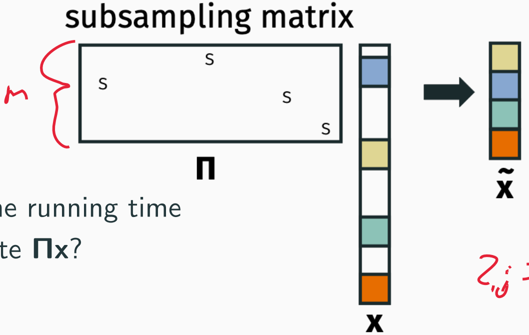
$$(1-\epsilon)\|\mathbf{x}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{x}\|_2^2 \leq (1+\epsilon)\|\mathbf{x}\|_2^2$$



We will learn about a truly brilliant method that runs in $O(n \log n)$ time. **Preview:** Will involve Fast Fourier Transform in disguise.

FIRST ATTEMPT

Let $\mathbf{\Pi}$ be a random sampling matrix. Every entry is equal to $s = \sqrt{n/m}$ with probability $1/n$, and is zero otherwise.



$\mathbb{E}[Z_{ij}] = \frac{1}{n}$

What's the running time to compute $\mathbf{\Pi x}$?

$Z_{ij} = 1$ if $\Pi_{ij} = s$

0 o.w.

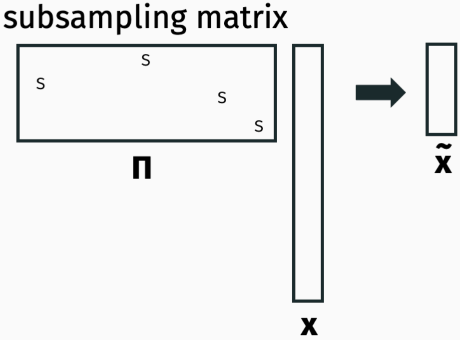
$\|\mathbf{\Pi x}\|_2^2 = \sum_{i=1}^{m} \sum_{j=1}^{n} Z_{ij}^2 \cdot s^2 x_i^2$

$\mathbb{E}[\|\mathbf{\Pi x}\|_2^2] =$ $m \cdot \frac{1}{n} \cdot s^2 \cdot \|x\|_i^2$

$= \|x\|_2^2$

FIRST ATTEMPT

So $\mathbb{E}\|\mathbf{\Pi x}\|_2^2 = \|\mathbf{x}\|_2^2$ in expectation. To show it is close with high probability we would need to apply a concentration inequality. How do you think this will work out?



VARIANCE ANALYSIS

$\|\mathbf{\Pi x}\|_2^2 = \sum_{i=1}^m \sum_{j=1}^n Z_i \cdot s^2 x_i^2$

$$\|x\|_p^p = \sum_{i=1} |x_i|^p$$

$$\begin{aligned}
\sigma^2 &= \mathrm{Var}[\|\mathbf{\Pi x}\|_2^2] \\
&= \sum_{i=1}^m \sum_{j=1}^n s^4 x_i^4 \,\mathrm{Var}[Z_i] \\
&= \frac{n^2}{m^2} \sum_{i=1}^m \sum_{j=1}^n \frac{1}{n} x_i^4 \\
&= \frac{n}{m^2} \sum_{i=1}^m \|x\|_4^4 = \boxed{\frac{n}{m}\|x\|_4^4}
\end{aligned}$$

VARIANCE ANALYSIS

$\|\mathbf{\Pi x}\|_2^2 = \sum_{i=1}^m \sum_{j=1}^n Z_i \cdot s^2 x_i^2$

$\sigma^2 \leq \frac{n}{m}\|x\|_4^4$

$\sigma = \sqrt{\frac{n}{m}}\, \|x\|_4^2$

$\|x\|_2^2$

Recall Chebyshev's Inequality:

$$\Pr[\left|\|\mathbf{\Pi x}\|_2^2 - \|\mathbf{x}\|_2^2\right| \leq 10 \cdot \sigma] \leq \frac{1}{100}$$

We want additive error $\left|\|\mathbf{\Pi x}\|_2^2 - \|\mathbf{x}\|_2^2\right| \leq \epsilon\|\mathbf{x}\|_2^2$

We need to choose m so that:

$\frac{n}{\sqrt{n}} < \varepsilon n$

$$10\sqrt{\frac{n}{m}}\|\mathbf{x}\|_4^2 \leq \epsilon\|\mathbf{x}\|_2^2.$$

$\frac{1}{\sqrt{n}} < \varepsilon$

$m > \frac{1}{\varepsilon^2}$

How do these two two norms compare?

$$\|\mathbf{x}\|_4^2 = \left(\sum_{i=1}^{n} x_i^4\right)^{1/2} \qquad\qquad \|\mathbf{x}\|_2^2 = \sum_{i=1}^{n} x_i^2$$

Consider 2 extreme cases:

$\|x\|_4^2 = \sqrt{n}$

$\|x\|_2^2 = n$

$\|x\|_4^2$

$= \|x\|_2^2$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad\qquad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

VARIANCE FOR SMOOOTH FUNCTIONS

We need to choose m so that:

$$\frac{1}{10}\sqrt{\frac{n}{m}}\|\mathbf{x}\|_4^2 \leq \epsilon\|\mathbf{x}\|_2^2.$$

Suppose $\mathbf{x}$ is very evenly distributed. I.e., for all $i \in 1, \ldots, n$,

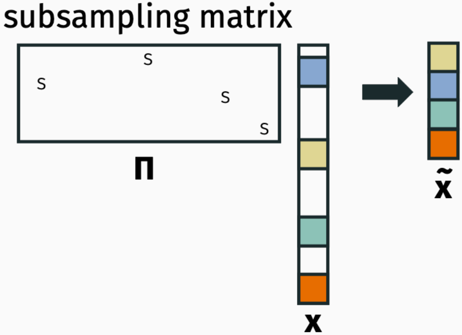
$$x_i^2 \leq \frac{c}{n}\sum_{i=1}^{n} x_i^2 = \frac{c}{n}\|\mathbf{x}\|_2^2$$

Claim: $\|\mathbf{x}\|_4^2 \leq \frac{c}{\sqrt{n}}\|\mathbf{x}\|_2^2$. So $m = O(c/\epsilon^2)$ samples suffices.[3]

[3] Using the right Bernstein bound we can prove $m = O(c\log(1/\delta)/\epsilon^2)$ suffices for failure probability δ.

VECTOR SAMPLING

So sampling does work to preserve the norm of $\mathbf{x}$, but only when the vector is relatively "smooth" (not concentrated). Do we expect to see such vectors in the wild?



Subsampled Randomized Hadamard Transform (SHRT) (Ailon-Chazelle, 2006)

$m \in \mathbb{R}^{n \times n}$

Key idea: First multiply $\mathbf{x}$ by a "mixing matrix" $\mathbf{M}$ which ensures it cannot be too concentrated in one place.

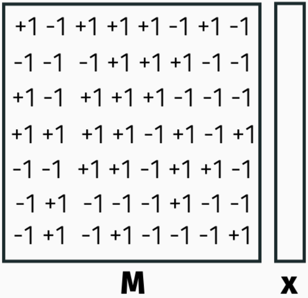
$\mathbf{M}$ should have the property that $\|\mathbf{M}\mathbf{x}\|_2^2 = \|\mathbf{x}\|_2^2$ exactly, or is very close. Then we will multiply by a subsampling matrix $\mathbf{S}$ to do the actual dimensionality reduction:

$$\mathbf{\Pi}\mathbf{x} = \mathbf{S}\mathbf{M}\mathbf{x}$$

Oh... and $\mathbf{M}$ needs to be fast to multiply by!

THE FAST JOHNSON-LINDENSTRAUSS TRANSFORM

Good mixing matrices should look random:



For this approach to work, we need to be able to compute $\mathbf{Mx}$ very quickly. So we will use a **pseudorandom** matrix instead.

Subsampled Randomized Hadamard Transform (SHRT) (Ailon-Chazelle, 2006)

$\mathbf{\Pi} = \mathbf{SM}$ where $\mathbf{M} = \mathbf{HD}$:

- $\mathbf{D} \in n \times n$ is a diagonal matrix with each entry uniform ± 1.
- $\mathbf{H} \in n \times n$ is a Hadamard matrix.

The Hadarmard matrix is an othogonal matrix closely related to the discrete Fourier matrix. It has two critical properties:

1. $\|\mathbf{Hv}\|_2^2 = \|\mathbf{v}\|_2^2$ exactly. Thus $\|\mathbf{HDx}\|_2^2 = \|\mathbf{x}\|_2^2$
2. $\|\mathbf{Hv}\|_2^2$ can be computed in $O(n \log n)$ time.

HADAMARD MATRICES RECURSIVE DEFINITION

Assume that n is a power of 2. For $k = 0, 1, \ldots,$ the k^{th} Hadamard matrix $\mathbf{H}_k$ is a $2^k \times 2^k$ matrix defined by:

$$H_0 = 1 \quad H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad H_2 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_k = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix}$$

The $n \times n$ Hadamard matrix has all entries as $\pm \frac{1}{\sqrt{n}}$.

Property 1: For any $k = 0, 1, \ldots$, we have $\|\mathbf{H}_k \mathbf{v}\|_2^2 = \|\mathbf{v}\|_2^2$ for all $\mathbf{v}$. I.e., $\mathbf{H}_k$ is orthogonal.

$H_{k-1} H_{k-1}^T = I$

$$H_k H_k^T = \frac{1}{2} \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix} \begin{bmatrix} H_{k-1} & H_{k-1} \\ H_{k-1} & -H_{k-1} \end{bmatrix}^T$$

$AA^T = I$

$$\begin{bmatrix} I & \\ & \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H_{k-1} \\ H_{k-1} \end{bmatrix} H_{k-1} + \begin{bmatrix} H_{k-1} \\ -H_{k-1} \end{bmatrix} H_{k-1}$$

$$\begin{bmatrix} I \\ 0 \end{bmatrix} \qquad \frac{1}{2}\left(\underbrace{H_{k-1} H_{k-1}^T + H_{k-1} H_{k-1}}\right) = I$$

$$H_{k-1} H_{k-1} - H_{k-1} H_{k-1} = 0$$

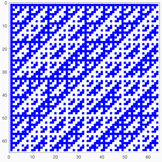
HADAMARD MATRICES

Property 2: Can compute $\mathbf{\Pi x} = \mathbf{SHDx}$ in $O(n \log n)$ time.

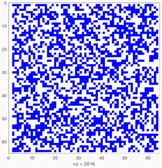
This is a nice exercise...can use recursion.

RANDOMIZED HADAMARD TRANSFORM

Property 3: The randomized Hadamard matrix is a good "mixing matrix" for smoothing out vectors.



Deterministic Hadamard matrix.



Randomized Hadamard **PHD**.

Fully random sign matrix.

Blue squares are $1/\sqrt{n}$'s, white squares are $-1/\sqrt{n}$'s.

RANDOMIZED HADAMARD ANALYSIS

Lemma (SHRT mixing lemma)

Let $\mathbf{H}$ *be an* $(n \times n)$ *Hadamard matrix and* $\mathbf{D}$ *a random* ± 1 *diagonal matrix. Let* $\mathbf{z} = \mathbf{H}\mathbf{D}\mathbf{x}$ *for* $\mathbf{x} \in \mathbb{R}^n$*. With probability* $1 - \delta$,

$$(z_i)^2 \leq \frac{c \log(n/\delta)}{n} \|\mathbf{z}\|_2^2$$

for some fixed constant c.

The vector is very close to uniform with high probability. As we saw earlier, we can thus argue that $\|\mathbf{S}\mathbf{z}\|_2^2 \approx \|\mathbf{z}\|_2^2$. I.e. that:

$$\|\mathbf{\Pi}\mathbf{x}\|_2^2 = \|\mathbf{S}\mathbf{H}\mathbf{D}\mathbf{x}\|_2^2 \approx \|\mathbf{x}\|_2^2$$

JOHNSON-LINDENSTRAUSS WITH SHRTS

Theorem (The Fast JL Lemma)

Let $\mathbf{\Pi} = \mathbf{SHD} \in \mathbb{R}^{m \times n}$ *be a subsampled randomized Hadamard transform with* $m = O\left(\frac{\log(n/\delta)\log(1/\delta)}{\epsilon^2}\right)$ *rows. Then for any fixed* $\mathbf{x}$*,*

$$(1-\epsilon)\|\mathbf{x}\|_2^2 \le \|\mathbf{\Pi x}\|_2^2 \le (1+\epsilon)\|\mathbf{x}\|_2^2$$

with probability $(1-\delta)$*.*

Very little loss in embedding dimension compared to full random matrix, and $\mathbf{\Pi}$ can be multiplied by $\mathbf{x}$ in $O(n \log n)$ (nearly linear) time.

$n d^2$

$n \log n d + d^3$

SHRT mixing lemma proof: Need to prove $(z_i)^2 \leq \frac{c \log(n/\delta)}{n} \|\mathbf{z}\|_2^2$ for all i.

Let $\mathbf{h}_i^T$ be the i^{th} row of $\mathbf{H}$. $z_i = \mathbf{h}_i^T \mathbf{D}\mathbf{x}$ where:

$$\mathbf{h}_i^T \mathbf{D} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \ldots & -1 & -1 \end{bmatrix} \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_n \end{bmatrix}$$

where $D_1, \ldots, D_n$ are random ± 1's.

This is equivalent to

$$\mathbf{h}_i^T \mathbf{D} = \frac{1}{\sqrt{n}} \begin{bmatrix} R_1 & R_2 & \ldots & R_n \end{bmatrix},$$

where $R_1, \ldots, R_n$ are random ± 1's.

RANDOMIZED HADAMARD ANALYSIS

So we have, for all i, $\mathbf{z}_i = \mathbf{h}_i^T \mathbf{D}\mathbf{x} = \frac{1}{\sqrt{n}} \sum_{i=1}^n R_i x_i$.

- $\mathbf{z}_i$ is a random variable with mean 0 and variance $\frac{1}{n}\|\mathbf{x}\|_2^2$, and is a sum of independent random variables.
- By Central Limit Theorem, we expect that:

$$\Pr[|\mathbf{z}_i| \geq t \cdot \frac{\|\mathbf{x}\|_2}{\sqrt{n}}] \leq e^{-O(t^2)}.$$

- Setting $t = \sqrt{\log(n/\delta)}$, we have for constant c,

$$\Pr\left[|\mathbf{z}_i| \geq c\sqrt{\frac{\log(n/\delta)}{n}}\|\mathbf{y}\|_2\right] \leq \frac{\delta}{n}$$

.

- Applying a union bound to all n entries of $\mathbf{z}$ gives the SHRT mixing lemma.

RADEMACHER CONCENTRATION

Formally, need to use Bernstein type concentration inequality to prove the bound:

Lemma (Rademacher Concentration)

Let $R_1, \ldots, R_n$ be Rademacher random variables (i.e. uniform ± 1's). Then for any vector $\mathbf{a} \in \mathbb{R}^n$,

$$\Pr\left[\sum_{i=1}^{n} R_i a_i \geq t\|\mathbf{a}\|_2\right] \leq e^{-t^2/2}.$$

This is call the Khintchine Inequality. It is specialized to sums of scaled ± 1's, and is a bit tighter and easier to apply than using a generic Bernstein bound.

FINISHING UP

With probability $1-\delta$, we have that all $\mathbf{z}_i \leq \sqrt{\frac{c\log(n/\delta)}{n}}\|\mathbf{c}\|_2$.

As shown earlier, we can thus guarantee that:

$$(1-\epsilon)\|\mathbf{z}\|_2^2 \leq \|\mathbf{S}\mathbf{z}\|_2^2 \leq (1+\epsilon)\|\mathbf{z}\|_2^2$$

as long as $\mathbf{S} \in \mathbb{R}^{m\times n}$ is a random sampling matrix with

$$m = O\left(\frac{\log(n/\delta)\log(1/\delta)}{\epsilon^2}\right) \text{ rows.}$$

$\|\mathbf{S}\mathbf{z}\|_2^2 = \|\mathbf{SHD}\mathbf{x}\|_2^2 = \|\mathbf{\Pi}\mathbf{x}\|_2^2$ and $\|\mathbf{z}\|_2^2 = \|\mathbf{x}\|_2^2$, so we are done.

JOHNSON-LINDENSTRAUSS WITH SHRTS

Theorem (The Fast JL Lemma)

Let $\mathbf{\Pi} = \mathbf{SHD} \in \mathbb{R}^{m \times n}$ *be a subsampled randomized Hadamard transform with* $m = O\left(\frac{\log(n/\delta)\log(1/\delta)}{\epsilon^2}\right)$ *rows. Then for any fixed* $\mathbf{x}$,

$$(1-\epsilon)\|\mathbf{x}\|_2^2 \leq \|\mathbf{\Pi x}\|_2^2 \leq (1+\epsilon)\|\mathbf{x}\|_2^2$$

with probability $(1-\delta)$.

Upshot for regression: Compute $\mathbf{\Pi A}$ in $O(nd \log n)$ time instead of $O(nd^2)$ time. Compress problem down to $\tilde{\mathbf{A}}$ with $O(d^2)$ dimensions.

$\tilde{O}(nd + d^3)$ vs $O(nd^2)$

BRIEF COMMENT ON OTHER METHODS

$\mathrm{nnz}(A) + d^{\omega}$ $\quad d \ll n$

$O(nd \log n)$ is nearly linear in the size of $\mathbf{A}$ when $\mathbf{A}$ is dense.

Clarkson-Woodruff 2013, STOC Best Paper: Possible to compute $\tilde{\mathbf{A}}$ with $\mathrm{poly}(d)$ rows in:

$$\underbrace{O\left(\mathrm{nnz}(\mathbf{A})\right)}_{} \text{ time.}$$

nd

- $\mathbf{\Pi}$ is chosen to be an ultra-sparse random matrix (spoiler: $\mathbf{\Pi}$ is count-sketch!).
- Uses totally different techniques (you can't do JL + ϵ-net).

Lead to a whole close of matrix algorithms (for regression, SVD, etc.) which run in time:

$$O\left(\mathrm{nnz}(\mathbf{A})\right) + \mathrm{poly}(d, \epsilon).$$